5.4 p. 296 PID} and Ring Maps M250 Burger

Example 8.

R a ring and $\{x_1, \dots, x_n\} \subseteq R$.
The ideal I generated by $\{x_1, \dots, x_n\}$

$$\langle x_1, \dots, x_n \rangle$$ is intersection

of All ideals of R containing $\{x_1, \dots, x_n\}$.

R com ring w/unity $\rightarrow$ $I = \{ a_1x_1 + \cdots + a_nx_n \mid a_i \in R \}$

Ex 9. In $\mathbb{Z}[x]$, $I = \{ a_0 + a_1x + \cdots + a_nx^n \mid a_0 \text{ is even} \}$
$= \langle 2, x \rangle$.

So How is $4x^7 + 13x^6 + 19x - 4$ in
$\langle 2, x \rangle$?

$$\langle 2, x \rangle = \{ a_1 \cdot 2 + a_2 \cdot x \mid a_1, a_2 \in \mathbb{Z}[x] \}$$

$$2a_1 + a_2 x =$$

$$\underbrace{2(-2)}_{\in \mathbb{Z}[x]} + \underbrace{(4x^6 + 13x^5 + 19)}_{\in \mathbb{Z}[x]} \cdot x \quad \checkmark$$

Defn 2 a principal ideal I has one generator $x \in I$ s.t.

$$\langle x \rangle = I.$$

Ex 10 $n\mathbb{Z} \subseteq \mathbb{Z}$

$$\underset{\langle n \rangle}{\overset{\|}{}}$$

$\langle 3 \rangle = \{ a_0 \cdot 3 \mid a_0 \in \mathbb{Z} \}$

Ex 11 Why is $I = \langle 2, x \rangle$ not principal?

If $I = \langle x \rangle$ then one could never get a constant term. If $I = \langle 2 \rangle$ then ~~have~~ All coefficients of every element in I are even, but $2x^2 + 3x + 8 \in I$ $\times$.

Ex 12 What if F is a field, then what about ideals in $F[x]$?

Defn. A PID is a $\underbrace{\text{Domain}}_{\text{no zero divisors}}$ in which every ideal is principal.

Important prop. not in text.

If F is a field then $F[x]$ is a P.I.D.

Pf. Let J be an ideal of $F[x]$.

Case 1 $J = \{0_F\}$ then $J = \langle 0 \rangle$ ✓.

Case 2 $J = F[x]$ then $J = \langle 1 \rangle$ ✓

Case 3 Now suppose $J \neq \{0_F\}$ ~~&~~ $\neq F[x]$.

$\rightarrow$ J has a non-zero element.

By well-ordering principal we can select the lowest degree non-zero element of J.

Let n = minimal degree of elements of J.

$n = 0$ $\rightarrow$ something like $5 \in J$ but $\frac{1}{5} \in F$ so $5 \cdot \frac{1}{5} = 1 \in J \rightarrow J = F[x]$.

so let $n > 0$

i.e. $n \geq 1$ so let $d(x) \in J$

s.t. $\deg d(x) = n$, this min. degree.

and let $f(x) \in J$. then by

Division ~~theorem~~ algorithm for polynomials

$$f(x) = d(x) \cdot q(x) + r(x) \quad , \quad q, r \in F[x]$$

where $r(x) = 0$ or $0 \leq \deg(r(x)) < n$

Since J is a ideal and $d(x) \in J$

then $d(x) \cdot q(x) \in J$. since $f(x) \in J$

$\longrightarrow$ $r(x) = f(x) - d(x)\,q(x) \in J$ ring!

But since $d(x)$ is minimal degree

$\longrightarrow$ $r(x) = 0$. $\therefore$ $f(x) = d(x) \cdot \underset{\in F[x]}{q(x)}$

hence $J = \langle d(x) \rangle$, a p. Ideal.

$\therefore$ $J \subseteq \langle d(x) \rangle$.

$\langle d(x) \rangle \subseteq J$

pf. this is obvious by the defn. of an ideal, since $d(x) \in J$, then outside multiplication stay in ~~$\langle d(x) \rangle$~~ J. a ring

4.

<u>Defn. 4</u> R_1, R_2 rings

$$f: R_1 \longrightarrow R_2$$

s.t. $f(a+b) = f(a) + f(b)$

$f(a \cdot b) = f(a) \cdot f(b)$

Is a '<u>ring homomorphism</u>'

f bijective $\rightarrow$ 'ring isomorphism'

<u>Note:</u> some defns require that if 1_{R_1} and 1_{R_2} exist then $f(1_{R_1}) = 1_{R_2}$.

Examples <u>not</u> in text.

<u>EX</u> $f: \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2$ $f(x) = x^2$

$f(x+y) = (x+y)^2 = x^2 + \underbrace{2xy}_{0} + y^2 = x^2 + y^2$

$= f(x) + f(y)$ ✓

$f(x \cdot y) = (xy)^2 = x^2 y^2 = f(x) \cdot f(y)$.

note. $f(1) = 1^2 = 1$ ✓

<u>EX.</u> Not a r.h.

$f: \mathbb{Z} \longrightarrow \mathbb{Z}$ $f(x) = 2x$

$f(x+y) = 2(x+y) = 2x + 2y = f(x) + f(y)$

but $f(xy) = 2xy \neq 2x \cdot 2y = 4xy$.

Also $f(1) = 2 \cdot 1 = 2 \neq 1$

Not in text

Lemma

$R \xrightarrow[\text{r.h.}]{f} S$

If R, S rings and f is a 'ring homomorphism' or 'ring map'.

then: (a) $f(0) = 0$

(b) $f(-r) = -f(r)$, $\forall r \in R$

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pf. (a) $f(0) = f(0+0) = \cancel{f(0) + f(0)}$

$\longrightarrow f(0) = 0$

(b) let $x \in R$, then $-x \in R$

and $f(x + -x) = f(x) + \underbrace{f(-x)}_{\text{add inverse}} = 0$

$f(0) =$

so $f(-x) = -f(x)$.

Ex. show $f(x) = 2x+5$ is Not a ring map $f: \mathbb{Z} \longrightarrow \mathbb{Z}$.

$f(0) = 2(0)+5 \neq 0$

done.

~~Given~~

Lemma

$$\begin{tikzcd} R \arrow[r, "f"] \arrow[rr, bend right] & S \arrow[r, "g"] & T \end{tikzcd}$$

Given f, g ring maps, then $(g \circ f)$ is a r-map.

Pf.

$$(g \circ f)(x+y) = g(f(x+y)) =$$

$$g(f(x) + f(y)) = g \circ f(x) + g \circ f(y)$$

since g is r-map.

$$(g \circ f)(x \cdot y) = g(f(x \cdot y)) = g(f(x) \cdot f(y))$$

$$= (g \circ f(x))(g \circ f(y)) \quad \checkmark$$

And if R, S, T have 1

then $(g \circ f)(1_R) = g(f(1)) = g(1_S)$

$$= 1_T.$$

$\therefore$ $g \circ f$ is a ring map $\square$

Defn. 5 Let $f: R_1 \xrightarrow[\text{homo.}]{\text{ring}} R_2$

The kernel of f

$\ker f = \{x \in R_1 \mid f(x) = 0_{R_2}\}$.

Prop 2 $f: R_1 \xrightarrow[\text{r. hom.}]{} R_2$ Then

$\ker f$ is an ideal of R_1.

Pf. 1st Need to show $\ker f$ is a subring of R_1.

Let $x, y \in \ker f$. Then $f(x+y) = f(x) + f(y) = 0 + 0 = 0$

$\therefore$ $x+y \in \ker f$

$f(x-y) = f(x) - f(y) = 0 - 0 = 0 \to x-y \in \ker f$

$x \in \ker f \to f(x) = 0$ but $f(-x) = -f(x) = -0 = 0$ by earlier lemma

$\therefore$ $-x \in \ker f$

Ideal Now let $a \in R_1$ and $x \in \ker f$.

Then $f(ax) = f(a) \cdot f(x) = f(a) \cdot 0 = 0 \to ax \in \ker f$ $\therefore$

$\ker f$ is an ideal of R_1 ∎